Journal of Sound and Vibration (1997) **204**(3), 556–559

A PHASE SPACE ANALYSIS OF A NON-LINEAR OSCILLATOR EQUATION

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(*Received* 18 *December* 1996)

In a recent paper, Hamdan and Shabaneh [1] investigated the large amplitude free vibrations of non-linear oscillators that can be modelled by the equation

$$\ddot{u} + mu + \varepsilon_1 u^2 \ddot{u} + \varepsilon_1 u \dot{u}^2 + \varepsilon_2 u^3 = 0, \tag{1}$$

where ε_1 and ε_2 are positive parameters, and m can take one of the three values $(-1, 0, 1)$. They constructed approximations to the analytic solutions of equation (1) by use of harmonic balance methods [2] and the time transformation procedure [3]. These results were then compared to numerical solutions of equation (1). The purpose of this note is to prove that *all* the solutions of equation (1) are periodic. This is accomplished by using methods from the qualitative theory of differential equations [2, 4, 5].

To proceed, change to phase space variables (x, y), where

$$x = u, \qquad y = \dot{u}. \tag{2}$$

In these variables, equation (1) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{(m + \varepsilon_1 y^2 + \varepsilon_2 x^2)x}{1 + \varepsilon_1 x^2}. \tag{3a, b}$$

The trajectories in phase space are then given by the solutions to the differential equation [2, 4, 5]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\left[\frac{m + \varepsilon_1 y^2 + \varepsilon_2 x^2}{1 + \varepsilon_1 x^2}\right]\left(\frac{x}{y}\right). \tag{4}$$

Inspection of equation (4) shows that it is invariant under the co-ordinate transformations

$$S_1\colon\ x \to -x, \quad y \to y \qquad \text{(reflection in the } y\text{-axis)},$$

$$S_2\colon\ x \to x, \quad y \to -y \qquad \text{(reflection in the } x\text{-axis)},$$

$$S_3\colon\ x \to -x, \quad y \to -y \qquad \text{(inversion through the origin)}.$$

Finally, equations (3) are a reversible system in the sense of Strogatz [6], i.e., these equations are invariant under the transformation

$$t \to -t, \qquad y \to -y. \tag{5}$$

The null-clines [2, 4] of equation (4) are those curves along which the slope $\mathrm{d}y/\mathrm{d}x$ of a trajectory in phase space is either zero or unbounded. Examination of equation (4) gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \infty\colon\ y = 0, \tag{6}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0\colon\ (m + \varepsilon_1 y^2 + \varepsilon_2 x^2)x = 0. \tag{7}$$

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Thus, equation (6) indicates that trajectories in phase space always cross the x-axis with infinite slope. From equation (7) it follows that the trajectories always cross the y-axis with zero slope. If $m = 0$ or 1, then this conclusion remains correct. However, if $m = -1$, then the trajectories have zero slope when they also cross the ellipse

$$\varepsilon_1 y^2 + \varepsilon_2 x^2 = 1. \tag{8}$$

Note that the null-clines intersect at the fixed points of the equation. (Fixed points are constant or equilibrium solutions to equations (3).) Also, the null-clines divide the phase plane into several open regions such that in each region the sign of $\mathrm{d}y/\mathrm{d}y$ is fixed. In these regions $\mathrm{d}y/\mathrm{d}x$ is bounded. The essential fact to bear in mind is that periodic solutions correspond to closed curves in phase space [2, 5–7].

With regard to equation (4), there are two cases to consider: case I corresponds to $m = 0$ or 1, $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$; case II corresponds to $m = -1$, $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$.

For case I, the x- and y-axes are null-clines. The slope, $\mathrm{d}y/\mathrm{d}x$, is zero along the y-axis and unbounded along the x-axis. The null-clines intersect at $(\bar{x}, \bar{y}) = (0, 0)$, the only fixed point for the system. The null-clines thus divide the phase plane into four open regions, as shown in Figure 1(a). The sign of $\mathrm{d}y/\mathrm{d}x$ in each of these regions is indicated by the symbols $(+)$ and $(-)$.

The analysis for this case begins with the selection of an arbitrary point on the positive y-axis. This is represented by the dot labelled A in Figure 1(a). The phase space trajectory that passes through this point is given in Figure 1(b). Note that the slope is zero at point A and is unbounded at point B where the trajectory intersects the x-axis. Applying the symmetry operation S_2 (reflection in the x-axis) to this curve gives Figure 1(c). Finally, the application of symmetry operation S_1 (reflection to the y-axis) gives the closed curve of Figure 1(d). The general conclusion reached from this geometrical analysis is that for

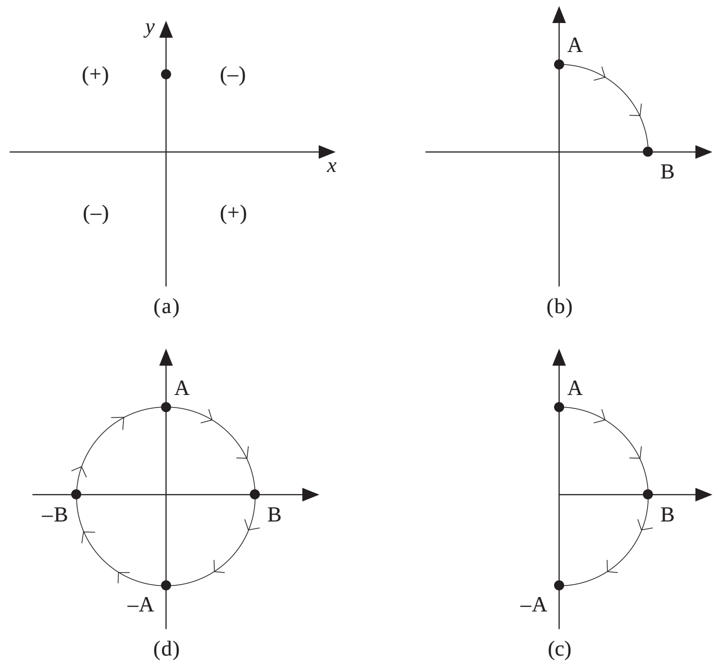
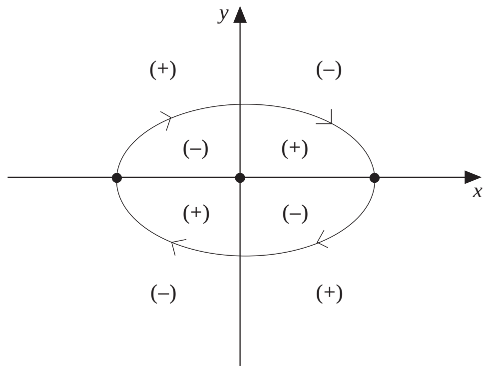


Figure 1. Case I, in which $m = 0$ or 1, $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$. (a) The signs of $\mathrm{d}y/\mathrm{d}x$; (b) typical trajectory in the first quadrant; (c) curve resulting from applying S_2 to (b); (d) application of S_1 to (c).



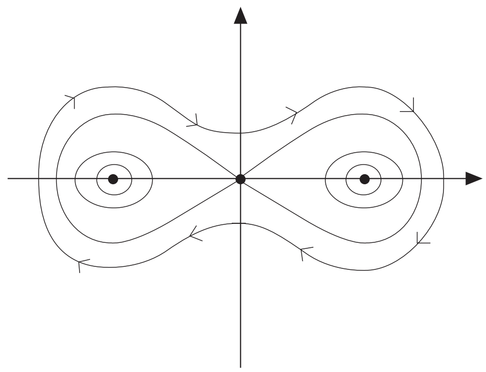


Figure 2. Case II, in which $m = -1$. (a) The signs of dy/dx and the three fixed points; (b) typical phase space curves.

$m = 0$ or 1, $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$, all the trajectories of equation (4) correspond to closed curves in the (x, y) phase space. Consequently, for case I, all solutions are periodic, except for the single fixed point at $(\bar{x}, \bar{y}) = (0, 0)$.

For case II, the x- and y-axes are still null-clines; i.e., the slope, dy/dx, is zero along the y-axis and unbounded along the x-axis. However, observe that dy/dx is *also* zero along the ellipse given by equation (8). Three fixed points now exist: they are located at

$$(\bar{x}_1, \bar{y}_1) = (-1/\sqrt{\varepsilon_2}, 0), \qquad (\bar{x}_2, \bar{y}_2) = (0, 0), \qquad (\bar{x}_3, \bar{y}_3) = (1/\sqrt{\varepsilon_2}, 0). \tag{9}$$

The null-clines thus divide the phase plane into eight open regions. These are shown in Figure 2(a), where the heavy dots indicate the three fixed points. The fixed points $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$ correspond to linear centers, while $(\bar{x}_1, \bar{y}_1)$ is a saddle point [8]. A theorem by Strogatz [6] on reversible systems can be applied to this problem to show that sufficiently close to the fixed points $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_3, \bar{y}_3)$ all the trajectories are closed curves. Using this result and constructing trajectories based on the properties of dy/dx, as indicated in Figure 2(a), it can be shown that typical trajectories have the structure as given in Figure 2(b). Thus, all the solutions for case II are periodic except for the three fixed points and the "figure-eight" curve which corresponds to two homoclinic orbits. These are trajectories in phase space that approach the origin as $t \to \pm\infty$. The reason that homoclinic orbits are not considered periodic solutions is that the "period" for motion on them is infinite [6].

In view of the above results, it can be concluded that all solutions to equation (1) are periodic except for the fixed points and the homoclinic orbits for the $m = -1$ case. (It is of course assumed that $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.) The procedure used to obtain this conclusion is based on the application of the qualitative theory of differential equations to our problem. This is a very powerful technique and can be directly applied to general second order differential equations that are currently used to model non-linear oscillatory systems [2]. Finally, the work of Hamdan and Shabaneh [1] is consistent with the results of this note.

ACKNOWLEDGMENT

This research was supported in part by research funds provided by DOE, NASA and MBRS Program at Clark Atlanta University.

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